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ON THE STABILITY OF THE PERMANENT ROTATIONS OF A RIGID BODY ABOUT A FIXED POINT IN BRUN'S PROBLEM†

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A criterion is proposed for the fixed sign character of inhomogeneous forms, and the identity of the sufficient and necessary conditions for the stability of the rotations of a heavy rigid body about the principal axis with the greatest moment of inertia is proved.

1. The following theorem is proved.

Theorem 1. The sufficient conditions for the stability of the permanent rotations of a rigid body about the smallest principal axis of inertia in Brun's problem are identical with the necessary conditions and have the form

$$C^2\omega^2 \geqslant \mu S^2, \quad S=\sqrt{A(C-B)}+\sqrt{B(C-A)}$$

The result obtained yields a softer stability condition than that obtained previously [1]. There is a proof of the strict inequality in [2].

The proof of the theorem rests on the criterion of the fixed sign character of the inhomogeneous forms

$$V(x)=V_{2\beta}(x_1,x_2,\ldots,x_n)+V_*(x) \tag{1.1}$$

$$x\in R^{n+l},\quad n\geqslant 1,\quad l\geqslant 1,\quad \beta\geqslant 1$$

where $V_{2\beta}(x_1, x_2, \ldots, x_n)$ is a fixed sign form of the smallest degree 2β of n variables and $V_*(x)$ is a polynomial of terms higher than a degree of 2β. It is obvious that the fixed sign character of $V(x)$ is equivalent to the absence of real solutions of the equation $V(x)=0$ in the neighbourhood of the origin of coordinates which are non-zero. Such solutions [3–5] may be sought in the form of parametric branches

$$x_j=\sum_{|p|=L}^{\infty} a_{jp}t^p,\quad j=1,2,\ldots,n \tag{1.2}$$

$$x_{n+1}=t_1^M,\ x_{n+2}=t_2^M,\ \ldots,\quad x_{n+l}=t_l^M$$

$$t=(t_1,t_2,\ldots,t_l),\quad t^p=t_1^{p_1}\times t_2^{p_2}\times\ldots\times t_l^{p_l}$$

$$a_{jp}\in R,\quad |p|=p_1+p_2+\ldots+p_l,\quad L>M$$

where p_j are non-negative integers.

Substituting (1.2) into (1.1), we obtain

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$$V(x(t)) = V_1(a_{jp}, t) = A_{Q(L)}(a_{jp}, t, L, M) + \dots \tag{1.3}$$

where $Q(L)$ is the degree of the form of the lowest order of the variables $t_1, t_2, \dots, t_l$ in the polynomial $V_1(a_{jp}, t)$. The value of M can initially be assumed to be equal to the least common multiple of the numbers $1, 2, \dots, 2\beta$ and then subsequently refined by contracting Q, L and M to their greatest common divisor D. The magnitude of L is determined during the construction of a branch of the solution of $V(x)=0$ from the condition

$$A_{Q(L)}(a_{jp}, t, L, M) = 0 \tag{1.4}$$

In the first step we put $L=M+1$. If a unique $a_{jp} \equiv 0$ ($j=1, 2, \dots, n$) is a solution of (1.4), then it is necessary to assume that $L=M+2$ and to continue the process of searching for an a_{jp} which is non-zero.

We reduce the fraction Q/M to the non-reducible fraction Q_1/M_1, and the following theorem then answers the question of the existence or non-existence of real branches in the case of $V(x)=0$.

Theorem 2. In the case when

1. (a) $Q_1=2\alpha+1$ (α is an integer) or (b) $Q_1=2\alpha$ and the form $A_{Q(L)}(a_{jp}, t, L, M)$ in the variables t_1, $t_2, \dots, t_l$ is of alternating sign for all real a_{jp} then the form $V_1(a_{jp}, t)$ is of alternating sign (real branches of $V(x)=0$ exist);

2. $Q_1=2\alpha$ and the form $A_{Q(L)}(a_{jp}, t, L, M) \gg 0$ for any $a_{jp} \in R$ then the form $V_1(a_{jp}, t)$ is of fixed sign (real branches of $V(x)=0$ do not exist);

3. $Q_1=2\alpha$ and the form $A_{Q(L)}(a_{jp}, t, L, M) \geqslant 0$ for all $a_{jp} \in R$ then the form $V_1(a_{jp}, t)$ may be of fixed sign or alternating sign which is established by taking into account terms in expansion (1.2) of an order higher than L and terms of an order higher than Q in (1.3) (real branches of the solution of $V(x)=0$ can exist).

The theorem is an extension of Theorem 1 in [6], where $n=t$ was a scalar variable. The fixed sign character of $V(x)$ follows from the fixed sign property of $V_1(a_{jp}, t)$ for all real $a_{jp} \in R$ while the sign alternation of $V(x)$ follows from the alternating sign character of $V_1(a_{jp}, t)$ on just one of the branches of (1.2).

In answering the question of the fixed sign character, it is desirable, in (1.2), to reduce the degree of M as far as possible, which facilitates the analysis of the expression $A_{Q(L)}(a_{jp}, t\, L, M)$ as a polynomial in t of a lower degree.

Theorem 3. In the analysis of the fixed sign character of the form (1.1) for the case when $\beta=1$, one can always assume that $M=1$ in expansion (1.2).

Proof. Let us assume that the M equals the least common multiple of (1.2) which is equal to 2. Since the expression $A_{Q(L)}(a_{jp}, t\, L, M)$ is determined by the polynomials $V_{2\beta}(x_1, x_2, \dots, x_n)$ and a certain part of the monomials $V_*(x)$ and the expansion $x_1, x_2, \dots, x_n$ begins with terms of the order of L, then, initially, $Q=2L$. We will next consider two possibilities: (1) $L=2\alpha$, and (2) $L=2\alpha+1$ (α is an integer).

For even L, the value of D = greatest common divisor of (M, L, Q) = greatest common divisor of $(2, 2\alpha, 4\alpha)=2$, and the reduced quantities $M_2=M/D=1$, $L_2=\alpha$, $Q_2=2\alpha$ are then the least possible values of the powers in (1.2) and the expression for the series (1.3). Consequently, for even L, one can immediately put $M=1$, thereby reducing the subsequent investigation to the use of Theorem 2.

In the case of odd L, the form (1.3) is of alternating sign (item 1(a) of Theorem 2), since, here, $Q/M=L=2\alpha+1$. Let us prove the alternating sign character of $V_1(a_{jp}, t)$ by directly putting $M=1$, in series (1.2). Since certain terms from $V_*(x)$ can have an influence on the structure of $A_{Q(L)}(a_{jp}, t, L, M)$, the relationship $Q=2L \in a_1L+a_2M$ holds for any monomial $x_1^{p1} \times x_2^{p2} \times \dots \times x_n^{pn} \times x_{n+1}^{q1} \times x_{n+2}^{q2} \times \dots \times x_{n+1}^{ql}$ with the notation $a_1=p_1+p_2+\dots+p_n$, $a_2=q_1+\dots+q$ of the polynomial $V_*(x)$. Here, the equality

$$2L = a_1L + a_2M \tag{1.5}$$

is satisfied in the case of monomials which make a contribution to the expression $A_{Q(L)}(a_{jp}, t, L, M)$.

We note that, when $a_1 \geqslant 0$, $a_2 \geqslant 0$, $L>M$, the latter equality only permits two values of a_1: (1) $a_1=1$ and (2) $a_1=0$. For odd values of L, the value $a_1=1$ is excluded from consideration since it follows from

(1.5) that $L=2a_2$ which does not hold for integer L and a_2. Consequently, $a_1=0$ corresponds to odd L. Since, $L/M=(2\alpha+1)/2$ in the case of the parametric solution, it is obvious that $L_1>\alpha$ when $M_1=1$. Putting $L_1=\alpha+1$ in the case of the monomials which satisfy equality (1.5), we find the estimate $Q_1=Q(\alpha+1)=a_1L_1+a_2$. From the identity $2Q_1=2L+a_1=2L$, in the case when $a_1=0$ which holds here, we obtain $Q_1=L=2\alpha+1$. In the given case $Q_1/M_1=2\alpha+1$ and according to item (2) of Theorem 2, we obtain the alternating sign form $V_1(a_{jp}, t)$. Since the investigation was carried out for all L, the theorem is proved.

2. We will now consider the proof of Theorem 1. The motion of a rigid body in Brun's problem is described by the Euler–Poisson equations [1, 2]. The first four integrals are known for them

$$V_0=Ap^2+Bq^2+Cr^2+\mu(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)=\text{const}$$

$$V_1=Ap\gamma_1+Bq\gamma_2+Cr\gamma_3=\text{const}$$

$$V_2=A^2p^2+B^2q^2+C^2r^2-\mu(BC\gamma_1^2+CA\gamma_2^2+AB\gamma_3^2)=\text{const}$$

$$V_3=\gamma_1^2+\gamma_2^2+\gamma_3^2=1$$

As the unperturbed motion, let us consider the permanent rotation of a body about the principal vertical axis with the greatest moment of inertia $(C>A,\ C>B)$

$$p=q=\gamma_1=\gamma_2=0,\quad r=\omega=\text{const},\quad \gamma_3=1 \tag{2.1}$$

The integrals of the equations of the perturbed motion, expressed in terms of the deviations from the solution of (2.1) $\varepsilon_1=p$, $\varepsilon_2=q$, $\varepsilon_3=r-\omega$, $\eta_1=\gamma_1$, $\eta_2=\gamma_2$, $\eta_3=\gamma_3-1$ are written in the form

$$V_{01}=A\varepsilon_1^2+B\varepsilon_2^2+C\varepsilon_3^2+\mu(A\eta_1^2+B\eta_2^2+C\eta_3^2)+2C\omega\varepsilon_3+2\mu C\eta_3=\text{const} \tag{2.2}$$

$$V_{11}=r_1+C\varepsilon_3\eta_3+C\varepsilon_3+C\omega\eta_3=\text{const}\qquad r_1=A\varepsilon_1\eta_1+B\varepsilon_2\eta_2$$

$$V_{21}=A^2\varepsilon_1^2+B^2\varepsilon_2^2+C^2\varepsilon_3^2-\mu(BC\eta_1^2+CA\eta_2^2+AB\eta_3^2)+2C^2\omega\varepsilon_3-2\mu AB\eta_3=\text{const}$$

$$V_{31}=k+\eta_3^2+2\eta_3=0\quad (k=\eta_1^2+\eta_2^2)$$

Theorem 1 will be proved if one succeeds in constructing a fixed sign function from (2.2) for the corresponding values of ω. With this aim, we shall reduce the number of variables to four by their successive elimination. From the last equation of (2.2), we find that

$$\eta_3=-1+(1-k)^{1/2}=-k/2-k^2/8-k^3/16-\ldots$$

and we substitute into the remaining integrals

$$V_{02}=A\varepsilon_1^2+B\varepsilon_2^2+\mu[(A-C)\eta_1^2+(B-C)\eta_2^2]+C(\varepsilon_3^2+2\omega\varepsilon_3)=\text{const}$$

$$V_{12}=r_1+C\varepsilon_3(1-k/2-k^2/8-\ldots)+C\omega(-k/2-k^2/8-\ldots)=\text{const}$$

$$V_{22}=A^2\varepsilon_1^2+B^2\varepsilon_2^2+\mu[B(A-C)\eta_1^2+A(B-C)\eta_2^2]+C^2(\varepsilon_3^2+2\omega\varepsilon_3)=\text{const}$$

Since the constant of the integral V_{12}, unlike that of V_{31}, is not fixed, we note that, according to Finsler's theorem [2], a fixed sign character of $V_{02}-\lambda_2V_{22}$ for the corresponding real λ_2 in the manifold $V_{12}=0$ is necessary and sufficient for the fixed sign property of the bunch of integrals $V_{02}-\lambda_2V_{22}-\lambda_1V_{12}$. Hence, on eliminating ε_3, from the integrals V_{02} and V_{22} using the equality $V_{12}=0$, we obtain the functions

$$W(\varepsilon,\eta)=V_{22}-CV_{02}=A(A-C)\varepsilon_1^2+B(B-C)\varepsilon_2^2+\mu(A-C)(B-C)k=\text{const}$$

$$W_0(\varepsilon,\eta)=V_{02}(\varepsilon_1,\varepsilon_2,\eta_1,\eta_2,\eta_3(\varepsilon,\eta))=\sum_{i=1}^{\infty} W_{2i}(\varepsilon_1,\varepsilon_2,\eta_1,\eta_2)$$

Here

$$W_2=AC\varepsilon_1^2+BC\varepsilon_2^2+\mu C[(A-C)\eta_1^2+(B-C)\eta_2^2]-2C\omega r_1+C^2\omega^2 k,$$

$$W_4=(C\omega k-r_1)^2,\quad W_6=kW_4,\ \ldots,\ W_{2j+2}=kW_{2j}$$

From these functions, we construct the bunch

$$K(\varepsilon,\eta,\lambda)=CW_0(\varepsilon,\eta)-\lambda W(\varepsilon,\eta)$$

for an arbitrary real value of λ.

Analysis shows that the quadratic form of the latter bunch degenerates when $\omega=\omega_*$, $\lambda=\lambda_*$ are such that

$$C^2\omega_*^2=\mu S^2,\quad \lambda_*=-1+k_1k_2$$

$$k_1=\sqrt{\frac{A}{C-B}},\quad k_2=\sqrt{\frac{B}{C-A}}$$

and takes the form

$$S^2(k_1\varepsilon_1-\sqrt{\mu}\eta_1)^2+S^2(k_2\varepsilon_2-\sqrt{\mu}\eta_2)^2$$

Since the quadratic part of the series $K(\varepsilon,\eta,\lambda_*)$ has a fixed sign when $\omega=\omega_*$ then, when investigating the fixed sign character of the bunch, it is necessary to take into account the terms $W_4(\varepsilon,\eta)$ and the criteria for the fixed sign character of inhomogeneous forms. On introducing the new variables $t_i=k_i\varepsilon_i-\sqrt{(\mu)}\eta_i$, $t_{i+2}=\eta_i$ $(i=1,2)$, we obtain $\varepsilon_i=(t_i+\sqrt{(\mu)}t_{i+2})/k_i$.

As a result of substituting ε_1, ε_2 into the expression $K(\varepsilon,\eta,\lambda_*)$, we reduce it to the form

$$K_2(t)=S^2(t_1^2+t_2^2)+(m_1t_3^2+m_2t_4^2+m_3t_1t_3+m_4t_2t_4)^2+\ldots$$

$$m_i=\sqrt{\mu}m_{i+2},\quad i=1,2;\quad m_3=\sqrt{B(C-A)}/S^{-1},\quad m_4=\sqrt{A(C-B)}/S^{-1}$$

where the dots denote terms of the sixth and higher orders.

Since $\beta=1$, $n=1=2$ in the case of the problem under investigation, then, by putting $M=1$ and $L=2$ according to Theorem 3, we can represent t_1 and t_2 in the form of series in the parameters t_3 and t_4

$$t_1=a_1t_3^2+a_2t_3t_4+a_3t_4^2+\ldots$$

$$t_2=b_1t_3^2+b_2t_3t_4+b_3t_4^2+\ldots$$

with unknown real coefficients a_j, b_j $(j=1,2,3)$. Here, the dots denote terms of higher than the second order in t_3 and t_4. As a result of substituting the latter parametrization into $K_2(t)$, we obtain the lowest order of the expansion $Q(L)=4$, and the corresponding mode of the lowest order takes the form

$$A_4(a_{jP},t,L,M)=(m_1t_3^2+m_2t_4^2)^2+S^2(a_1t_3^2+a_2t_3t_4+a_3t_4^2)^2+S^2(b_1t_3^2+b_2t_3t_4+b_3t_4^2)^2 \tag{2.3}$$

In order to apply Theorem 2, we shall seek real a_j, b_j $(j=1,2,3)$ which transforms the form A_4 into zero when $t_3^2+t_4^2>0$. Since the latter is the sum of squares, this is equivalent to the solution of the system obtained by equating the expressions in parentheses on the right-hand side of (2.3) to zero. Since, $m_1>0$,

$m_2 > 0$, this system is only consistent when $t_3 = t_4 = 0$. With the requirement that $t_3^2 + t_4^2 > 0$ and the a_j, b_j ($j = 1, 2, 3$) should be arbitrary and real, we have $A_4 \geqslant 0$. Hence, the form $A_4(a_j, b_j; t_3, t_4; 2; 1)$ is positive definite. By Theorem 2, the form $K(\varepsilon, \eta, \lambda_*)$ is also positive definite.

Consequently, Theorem 1 is proved.

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